Optimizing Retirement Income & Annuities

FINE4060: Advanced Personal Finance

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Lect. #8: [28 March 2019]

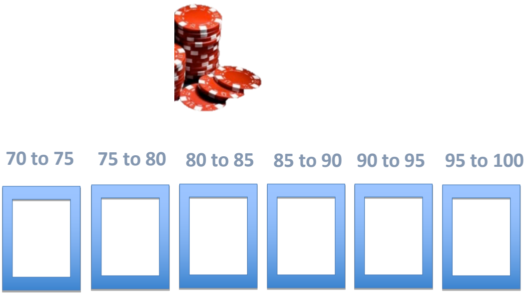
Technical Objectives for today.

In this lecture I will discuss various topics related to *optimizing* retirement income in the presence of longevity risk. In particular, by the end of today's lecture you should be able to answer the following questions.

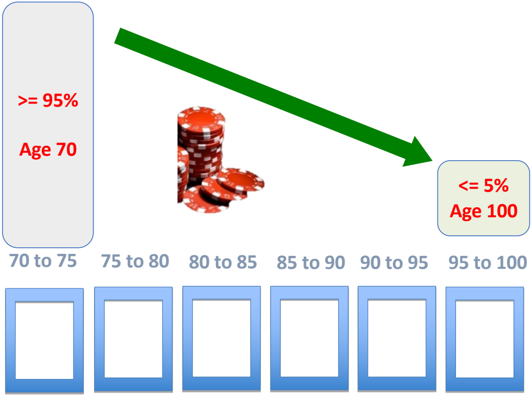
- How do I rationally smooth consumption towards the end of the lifecycle, when I don't really know how long I'm going to live?
- What is the optimal drawdown (a.k.a. spending, consumption) rate and what does it depend on?
- In an early lecture we discussed the *infamous* 4% rule of retirement spending. Is there any theoretical justification for this strategy?
- How do pension (life) annuities *qualitatively* influence the optimal drawdown strategy?

Play a Game with Retirement Chips

You have 60 poker chips. They are your retirement *nest egg*. Allocate them across the boxes. There is no right/wrong answer. It's preferences.



But, don't forget longevity risk



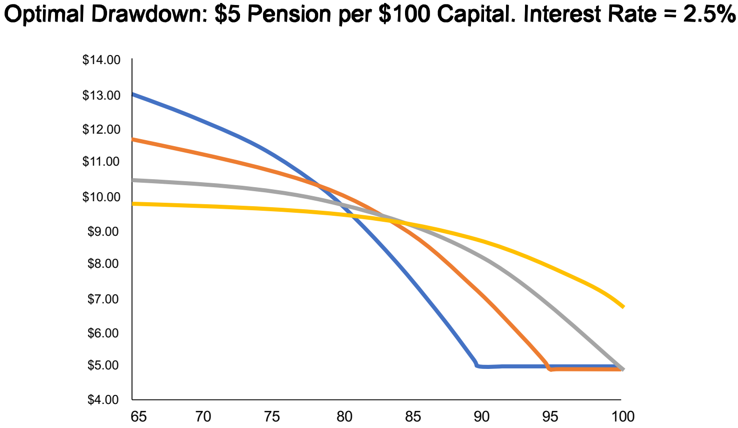
FYI only. This was the average across many experiments

To Ponder

How would your answer change if...

- Chips grow and earn guaranteed interest in the boxes, while you are waiting to (get there and) spend them.
- You are entitled to a pension (life) annuity of 10 chips per box.
- When you die, all the unused (unspent) chips go to your children.

Consistent with the LCM: Which I'll get back to later...



The first economist to *really* think about LCM issues...

Irving Fisher (1867-1947)

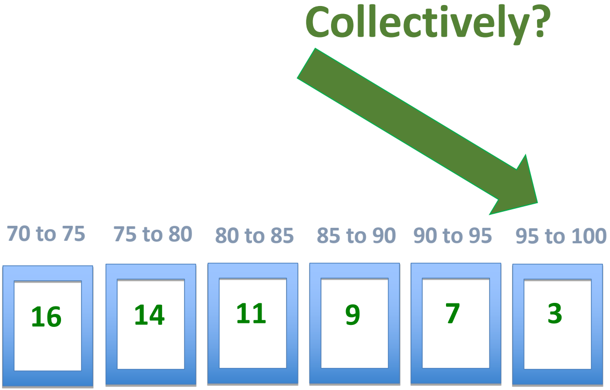
- **Professor of Economics, Yale.**
- **Created first inflation-indices.**
- **Inventor, entrepreneur, spokesperson, health advocate.**
- **Infamously incorrect forecast of the stock market in 1929.**

Lifecycle Model (LCM): Still relevant after all these years

As far as I am aware, no one has challenged the view that if people were capable of it, they ought to plan their consumption, saving and retirement according to the principles enunciated by [the lifecycle model of] Modigliani and Brumberg in the 1950s.

Prof. Angus S. Deaton
Princeton University
Nobel Laureate 2015

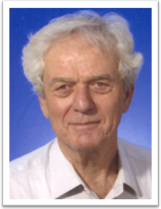
But once, the game got weird...



Pooling credit goes to...

Irving Fisher (1867-1947)

- Professor of Economics, Yale.
- Created first inflation-indices.
- Inventor, entrepreneur, spokesperson, health advocate.
- Infamously incorrect forecast of the stock market in 1929.



Original work by Irving Fisher refined by Professor Menachem Yaari in 1965, also at Yale University (at the time.)

The Diversity of Retirement Spending Rates in Canada

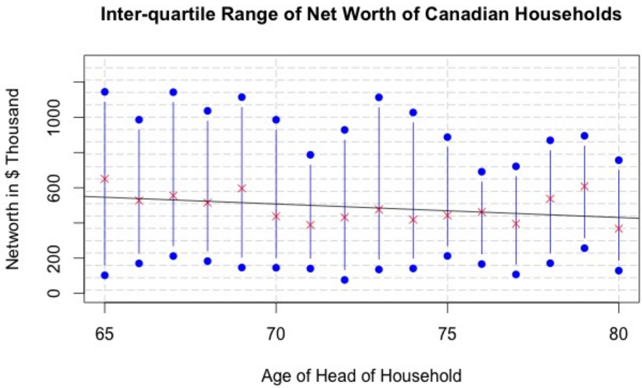


Figure: *Source: Statistics Canada. Survey of Financial Security 2012. Sum of the market value of 10 financial categories plus 5 non-financial categories minus the value of 7 debt categories. Sample weighted.*

So, wealth declines (slowly) in Retirement

In a (very) simple regression with age x_i as the *independent* variable and net-worth w_i as the *dependent* variable:

$$w_i \;=\; \alpha_0 \;+\; \alpha_1 \, x_i \;+\; e_i,$$

net-worth declines by approximately $\alpha_1 = -\$18,000$ per year for the 3,179 sampled households at or above the age of 65 in the Canadian SFS2012 dataset. **Surprise:** Financial assets (a.k.a. investment accounts) decline by **approximately 4%** of age-65 wealth per year.

There is quite a bit of diversity in retirement spending rates

Change in family wealth based on age of oldest person in household.

Age	75th Percentile.	Median	25th Percentile.
70	-4.35%	-10.48%	+7.45%
75	-8.34%	-6.77%	+4.70%
80	-17.20%	-10.64%	+14.49%

Source: SFS 2012. Sample weighted. Two year window.
Average change between age 70 and 80 is: -3.33%.

Note the wide dispersion in spending rates. **Note:** At the age of 70 approximately 1/3 of Canadian households have a negative spending rate (i.e. they continue to save). Heterogeneity is consistent with U.S. data.

Question: What objective are people trying to achieve?

Lifecycle Model (LCM) of consumption smoothing

Maximize the expected discounted *lifetime utility* of consumption spending:

$$\int_x^{\omega} e^{-\rho(a-x)}\, p_x(a)\, u(c_a)\, da, \tag{1}$$

where x denotes current age, ρ is a subjective discount rate, $p_x(a)$ denotes the survival probability to chronological age a, $u(.)$ is a utility function (more to come), c_a is the consumption rate, and ω is the maximum age.

FYI, this will result in a flat (constant) consumption rate (e.g. \$18,000 per year) when ρ is equal to the valuation rate (r), and the probability of survival to age ω, is $p_x(\omega) = 100\%$.

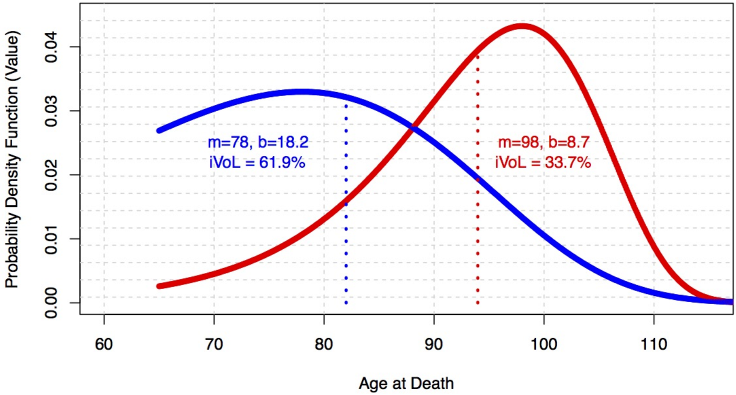
Why the dispersion in spending (a.k.a. drawdown) rates?

Within a lifecycle (LCM) framework, what might explain the ("rational") reason some retirees spend (much) more than others?

1. **Leisure, Labor & Legacy Preferences:** Discuss...
2. **Portfolio Choice, Markets & Investment Views:** Discuss...
3. **Longevity & Mortality Expectations:** Discuss...

Today: Focus on Impact of Lifetime Uncertainty

Two people enter retirement with the same wealth, identical risk and bequest preferences, but different longevity prospects. How will their ("rational") spending differ? And, how do life annuities (pooling) fit-in?

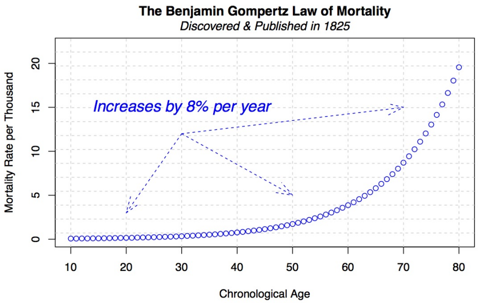


Back to the Benjamin Gompertz Law of Mortality

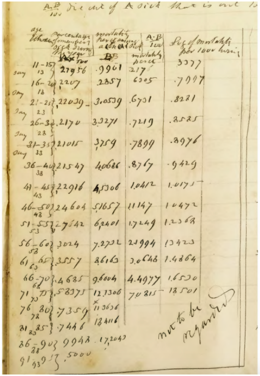
Recall that under the Gompertz (no Makeham term) law of mortality.

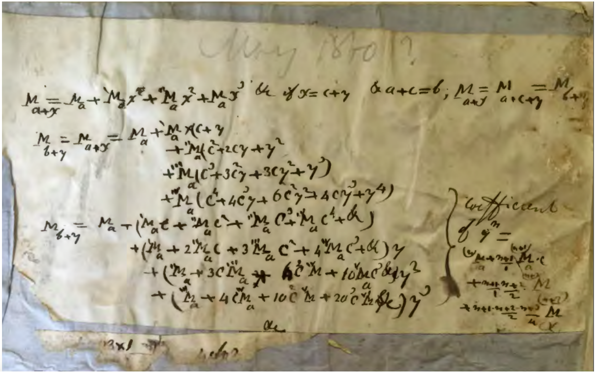
$$\lambda_x = \frac{1}{b} e^{(x-m)/b} = \left(\frac{e^{-m/b}}{b} \right) e^{\frac{1}{b}x} = h e^{gx}$$

where x is chronological age, m is the modal value of life in years (e.g. 98) and b is the dispersion parameter in years (e.g. 8.7). Alternatively, h is the initial mortality rate (IMR) and g is the mortality growth rate (MGR).



Side Note: Benjamin Gompertz, in his own words...





Source: Archives of the British Institute of Actuaries, London. Collection of documents from the estate of Benjamin Gompertz, circa 1820 to 1840.

The (Retirement) Lifecycle Model with Gompertz Mortality

- Maximize discounted utility of consumption over remaining lifetime:

$$J = \max_{c_s} E\left[\int_0^\infty e^{-\rho s} u(c_s) 1_{\{s \le \zeta\}} ds\right], \tag{2}$$

where: $\zeta < \infty$ is the (random) remaining lifetime satisfying $\Pr[\zeta \ge s] = p_x(x+s)$. **Note:** The extra step is a formality to keep the mathematicians happy.

- When the mortality rate λ_t is deterministic (i.e. no randomness in Biological age), optimal consumption c_s^* and $1_{\{s \le \zeta\}}$ are independent:

$$J = \max_{c_s} \int_0^{\omega - x} e^{-\rho s} u(c_s) p_x(x+s) ds.$$

- For most of what follows, $u(c) = c^{(1-\gamma)}/(1-\gamma)$, which is constant relative risk aversion (CRRA) utility, and γ is the key coefficient.

Now, remember, there is also a budget constraint

- In the most general form:

$$F'_s = (r + \lambda_s)F_s + \pi_s - c_s,$$

  where $F_0 = w > 0$, $F_{(\omega - x)} = 0$ (i.e. no bequest motive); π_s is existing pension income, r is the risk-free rate and λ_s are mortality credits, if pooled. **Note:** Stop, discuss and explain.
- In what follows, I'll (only) assume the budget constraint is:

$$F'_s = rF_s + \pi_0 - c_s.$$

- In English (not Greek), the change in your wealth $F'_s \approx \Delta F_s$, is equal to the sum of the interest your earn on wealth $rF_s\Delta t$, plus the pension annuity income $\pi_0 \Delta t$ minus the consumption $c_s \Delta t$.
- Don't lose sight of the objective: We are solving for the optimal c^*_s.

Analytic solution to the optimal consumption rate

The optimal (wealth during retirement) function F_s satisfies a second-order non-homogenous differential equation in regions where $F_s \neq 0$.

(*Note: At this point you will have to trust me on this one.*)

The ordinary differential equation (ODE) is:

$$F_s'' - \left(\frac{r - \rho - \lambda_s}{\gamma} + r \right) F_s' + r \left(\frac{r - \rho - \lambda_s}{\gamma} \right) F_s = - \left(\frac{r - \rho - \lambda_s}{\gamma} \right) \pi_0.$$

In general the ODE can't be solved explicitly unless λ_s is constant (i.e. no aging). However, one can express the relevant (consumption) function analytically under the (yes!) Gompertz mortality assumption.

Finally: The Analytic Solution

- Generally speaking there are two qualitatively different cases; with and without pension annuity income.
- First, when $\pi_0 = 0$ (i.e. no pension), the optimal consumption is:

$$c_s^* = c_0^* e^{ks} \left(p_x(x+s)\right)^{1/\gamma}, \tag{3}$$

where the new constant $k := (r-\rho)/\gamma$.

- The optimal trajectory of wealth F_s^* (noting that $F_0 = w$) is:

$$F_s^* \left(w - c_0^* a_x^s(r-k, m^*, b)\right) e^{rs}, \tag{4}$$

where $m^* = m + b\ln[\gamma]$, and $a_x^s(.,.,.)$ is a temporary life annuity. Remember `GTLA`. **Note: There is a lot going on here.**

- And, the initial consumption (spending) rate is:

$$c_0^* \;=\; \frac{w}{a_x^\omega(r-k, m^*, b)} \;=\; \frac{w}{a_x(r-k, m^*, b)}. \tag{5}$$

Code-up in R

The optimal consumption (i.e. spending) function at any time `s`, during retirement is denoted by `CRET`, and depends on initial wealth: `w`, initial age: `x`, real interest (valuation) rate: `r`, Gompertz parameters: `m,b`, the coefficient of relative risk aversion: `gam` the subjective discount rate: `rho`, which for the most part is assumed equal to: `r`.

Script

```
CRET<-function(w,x,s,r,m,b,gam,rho){
  k<-(r-rho)/gam
  mstar<-m+b*log(gam)
  c0<-w/GILA(x,r,mstar,b)
  c0*exp(k*s)*TPXG(x,s,m,b)^(1/gam)
}
```

Note the 8 (eight) different parameters and what they represent.

Numerical Examples: m=89.335, b=9.5, w=100

Let's modify the valuation rate r, which I assume is set equal to the subjective discount rate rho, and see how the consumption changes:

Command Line

```
> round(CRET(100,65,0,0.005,89.335,9.5,4,0.005),2)
[1] 3.33
> round(CRET(100,65,0,0.01,89.335,9.5,4,0.01),2)
[1] 3.63
> round(CRET(100,65,0,0.02,89.335,9.5,4,0.02),2)
[1] 4.27
> round(CRET(100,65,0,0.04,89.335,9.5,4,0.04),2)
[1] 5.69
```

These numbers are in dollars per initial $100 of retirement wealth, which can also be interpreted as a spending rate. **Intuitively:** You can drawdown more if interest rates are higher.

Numerical Examples: `m=89.335, b=9.5, w=100`

Let's modify the coefficient of relative risk aversion `gam`, or γ in the derivations, and see how the consumption changes:

Command Line

```
> round(CRET(100,65,0,0.04,89.335,9.5,1,0.04),2)
[1] 7.4
> round(CRET(100,65,0,0.04,89.335,9.5,4,0.04),2)
[1] 5.69
> round(CRET(100,65,0,0.04,89.335,9.5,10,0.04),2)
[1] 5.11
> round(CRET(100,65,0,0.04,89.335,9.5,100,0.04),2)
[1] 4.42
> round(CRET(100,65,0,0.04,89.335,9.5,1000,0.04),2)
[1] 4.17
```

Intuitively: The more (longevity) risk averse you are, the less you spend.

Numerical Examples: `m=89.335, b=9.5, w=100`

Modify the value of `s`, to see how the consumption changes over time. Retire at age `x=65` and this function (also) tells you how to adjust.

Command Line

```
> round(CRET(100,65,0,0.04,89.335,9.5,3,0.04),2)
[1] 5.95
> round(CRET(100,65,10,0.04,89.335,9.5,3,0.04),2)
[1] 5.67
> round(CRET(100,65,20,0.04,89.335,9.5,3,0.04),2)
[1] 4.94
> round(CRET(100,65,35,0.04,89.335,9.5,3,0.04),2)
[1] 2.19
```

Intuitively: As time goes on, you rationally plan to spend less. (Remember the chips!)

A Table of Numerical Values

Optimal Consumption Rate				
Coefficient of Relative Risk Aversion (CRRA) $\gamma = 4$				
Initial Wealth of $100 Invested at Following REAL Rates...				
	$r = 0.5\%$	$r = 1.5\%$	$r = 2.5\%$	$r = 3.5\%$
Age 65	$3.330	$3.941	$4.605	$5.318
5 Years Later	$3.286	$3.888	$4.544	$5.247
10 Years Later	$3.212	$3.801	$4.442	$5.130
20 Years Later	$2.898	$3.429	$4.007	$4.627
30 Years Later	$2.156	$2.552	$2.982	$3.444

Note that $\rho = r$, which in the absence of mortality and longevity risk would lead to a flat (constant) consumption profile over the entire lifecycle. $(m = 89.335, b = 9.5)$.

Next case: When $\pi \neq 0$ and you have pension income

- When $\pi_0 > 0$ (i.e. pre-existing pension), the optimal consumption function looks quite similar:

$$c_s^* = c_0^* e^{ks} \big(p_x(x+s) \big)^{1/\gamma}, \tag{6}$$

but is only valid until a **wealth depletion time** (WDT) $\tau \leq \omega - x$. After that time, $c_\tau = \pi_0$ and you *live* on your pension income π_0.

- In this case ($\pi_0 > 0$), the initial consumption (spending) rate is:

$$c_0^* \;=\; \frac{(w + \pi_0/r)e^{r\tau} - \pi_0/r}{a_x^\tau(r-k, m^*, b)e^{r\tau}}, \tag{7}$$

where a_x^τ is the `GTLA` function in R. So (a bit messy, but) you have to solve for τ first, and then plug-into the above and retrieve c_0^*.

- **Note:** If you survive to time τ, you (rationally) run out of money!

This is complicated, so I'll start with an example: If you have pension annuity income, life is different

The greater your pension income π_0 as a fraction of your investable wealth w, the more you are willing to spend from it.

Optimal Initial Withdrawal Rate from \$100				
As a Function of Pension Income π_0				
Depending on the Coefficient of Relative Risk Aversion				
	$\gamma = 1$	$\gamma = 2$	$\gamma = 4$	$\gamma = 8$
No Pension	6.330%	5.301%	4.605%	4.121%
$\pi_0 = \$1$	6.798%	5.653%	4.873%	4.324%
$\pi_0 = \$2$	7.162%	5.924%	5.078%	4.480%
$\pi_0 = \$5$	8.015%	6.553%	5.551%	4.839%
Note: Gompertz Mortality ($m = 89.3$, $b = 9.5$) and $r = 2.5\%$				

Note that $\rho = r$ (again.)

Deriving optimal (c^*, F^*, τ) gets messy and we patented it.

US008781937B2

(12) **United States Patent**
Milevsky et al.

(10) **Patent No.: US 8,781,937 B2**
(45) **Date of Patent: Jul. 15, 2014**

(54) **OPTIMAL PORTFOLIO WITHDRAWAL DURING RETIREMENT IN THE PRESENCE OF LONGEVITY RISK**

(75) Inventors: **Moshe A Milevsky**, Toronto (CA); **Huaxiong Huang**, Markham (CA)

(73) Assignee: **Qwema Group, Inc.**, Toronto (CA)

(*) Notice: Subject to any disclaimer, the term of this patent is extended or adjusted under 35 U.S.C. 154(b) by 0 days.

(21) Appl. No.: **13/602,996**

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CPC ***G06Q 40/08*** (2013.01); ***G06Q 40/06*** (2013.01)
USPC .... **705/36 R**; 705/4

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See application file for complete search history.

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(Continued)

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(57) **ABSTRACT**

A method, system, and medium for recommending an optimal withdrawal amount, for a given period, from a retiree's portfolio accounts comprised of relatively risky and relatively safe financial assets used to finance retirement. The user supplies information about the retiree's personal characteristics, including age, gender, and health status. Details of the retiree's financial situation are also supplied, including the retiree's total liquid wealth, the current value of relatively risky and relatively safe assets, and any after-tax pension and other annuity income. Risk (standard deviation of return),

The secret sauce: Wealth Depletion Time (WDT) τ

Consider the function $f(t)$ defined by the following **R** script or code:

Script

```
f<-function(t){((((GILA(x,r,m,b)*(1-psi)/psi+1/r)*
exp(r*t)-1/r)*TPXG(x,t,m,b)^(1/gam))/(exp(r*t)*
GTLA(x,x+t,r,m+b*log(gam),b))-1}
```

where `GILA(.)` denotes the Gompertz immediate annuity factor, `GTLA(.)` denotes the Gompertz temporary annuity factor, `TPXG(.)` is the Gompertz survival probability. Note the explicit variables (x, m, b), as well as the real valuation rate r, the coefficient of (longevity) relative risk aversion γ, and the (new) variable $\psi := a_x\pi_0/(w + a_x\pi_0)$ which measures the fraction of the balance sheet that is pre-pensionized. Explanation to come soon...

Computing the WDT

The value of τ for which $f(\tau) = 0$ is the wealth depletion time (WDT), and use the built-in one-dimensional solver in **R** to obtain numerical values.

Script

```
x<-65; r<-0.025; b<-9.5; m<-89.335;
wdt<-matrix(0,nrow=100,ncol=8)
for (j in 1:8){
  gam<-j
for (i in 1:99){
  psi<-i/100
  wdt[i,j]=uniroot(f,lower=0,upper=100)$root
}}
```

Note that this can be rather slow, b/c we are valuing the annuity at every round. If speed is of the essence, then the procedure can be made more efficient by storing `GILA` and `GTLA` values.

Numerical example of the wealth depletion time...

The wealth depletion age is $x + \tau$, where τ is the WDT. I report values for $\psi = 50\%$, that is when 50% of the balance sheet is pensionized and when $\psi = 75\%$, for values of $\gamma = 1, 4, 8$.

Script

```
> wdt[50,8]+65
[1] 104.0597
> wdt[50,4]+65
[1] 98.60194
> wdt[50,1]+65
[1] 88.50772
> wdt[75,8]+65
[1] 96.9291
> wdt[75,4]+65
[1] 91.95195
> wdt[75,1]+65
[1] 83.08823
```

Interpretation and Detailed Case Study

You are $x = 65$ years old, have $w = 500,000$ in liquid (investable) wealth and are entitled to a pension annuity of $\pi_0 = 31,675$ per year for life. The actuarial present value of your pension annuity, at $r = 2.5\%$ is:

Command Line

```
> 31675*GILA(65,0.025,89.335,9.5)
[1] 500069.3
```

So, add the two numbers together, that is: $(w + a_x\pi_0)$, and your total economic balance sheet is \$1 million, of which 50% is pensionized. Note that under these parameters, the probability you live to age 100 is:

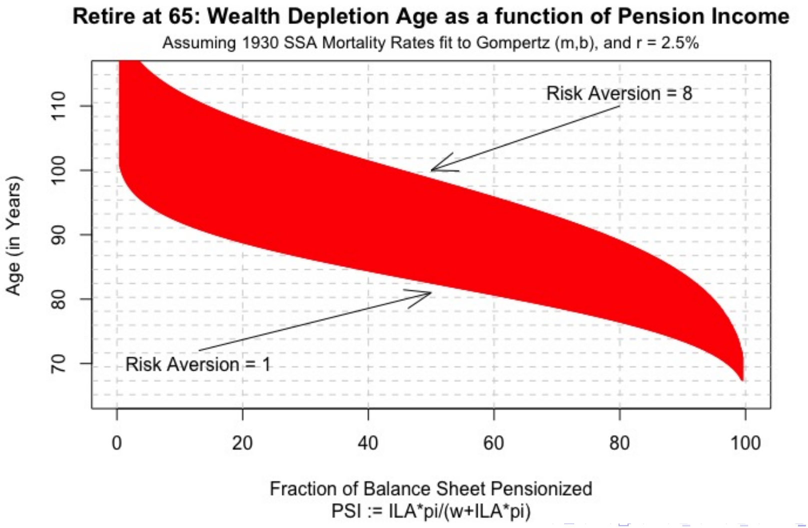
Command Line

```
> round(TPXG(65,35,89.335,9.5),5)
[1] 0.05
```

So, at what age will you "rationally" deplete wealth?

Visualizing the WDT

The greater your pension income, the earlier you deplete investable wealth.



So, the R-code for the optimal spending rate is...

Command Line

```
m<-89.335; b<-9.5; r<-0.025; x<-65;
w<-100; pi<-5; gam<-4; s<-0
psi<-pi*GILA(x,r,m,b)/(w+pi*GILA(x,r,m,b))
f<-function(t){(((GILA(x,r,m,b)*(1-psi)/psi+1/r)*
exp(r*t)-1/r)*TPXG(x,t,m,b)^(1/gam))/(exp(r*t)*
GTLA(x,x+t,r,m+b*log(gam),b))-1}
WDT<-uniroot(f,lower=0,upper=100)$root
mstar<-m+b*log(gam)
c0<-((w+pi/r)*exp(r*WDT)-pi/r)/
(GTLA(x,x+WDT,r,mstar,b)*exp(r*WDT))
round((c0*TPXG(x,s,m,b)^(1/gam)-pi)/w,4)
[1] 0.0555
```

Spend 5.55% of $w = \$100$ nest egg, for a total of $c^* = \$10.55$ at age 65.

Finally: The Seven Stage of Retirement Income Awareness

Question: How much money should you withdraw from your portfolio?

1. $x = 4\%$ of your *original* nest egg, adjusted for inflation. (Bengen.)
2. $x = 4\%$ of the *current* value of your portfolio (*Not* Bengen.)
3. $x \neq 4\%$, b/c "times have changed" since 90s. (Modern Bengen.)
4. $x = f(a)$, which is a function of your current (and true) age, a.
5. $x = f(r, a)$, where r denotes current interest rate levels.
6. $x = f(r, a, \gamma)$, which incorporates personal risk aversion, γ.
7. $x = f(r, a, \gamma, \pi)$, accounting for pre-existing pension income π.

It's a Wrap!

We are done...