Measuring and Modeling Portfolio Longevity

FINE4060: Advanced Personal Finance

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Main Question for Today

I am nearing retirement (age 65, for example) and have \$1,000,000 invested in a portfolio of stocks & bonds and would like to withdraw \$40,000 per year, adjusted for inflation, for the remaining of my life. (Think a smooth and constant consumption rate.)

How long will the money last? (In practice, not in theory.) Should I be planning to spend less (or maybe more) than this number? Is \$40,000 per year (i.e. 4% of the original \$1,000,000) a realistic spending rate?

Why 4%?

Vanguard

Revisiting the '4% spending rule'

Vanguard research

August 2012

Proposed by Bill Bengen in the early 1990s

New Math for Retirees and the 4% Withdrawal Rule

Retiring
By TARA SIEGEL BERNARD MAY 8, 2015

Bill Bengen, a retired financial planner, at home in La Quinta, Calif. He founded the 4 percent rule of annual withdrawals from a retirement account. Jaime Kowal for The New York Times

Everyone have an opinion about it?


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Say Goodbye to the 4% Rule

If the conventional wisdom no longer holds about spending in retirement, what are the alternatives? Here are three of them.

I mean everyone...


The Motley Fool

Stocks How to Invest Retirement Personal Finance Community

4 Serious Problems With the 4% Retirement Rule

The 4% rule of retirement is one of the most frequently cited guidelines for retirees to determine a safe withdrawal rate from their savings. The basic concept is that if you withdraw 4% of your retirement savings during your first year of retirement and increase this amount in subsequent years to keep up with inflation, you shouldn't run out of money during your retirement.

AUTHOR

Matthew Frankel

Second Motivation: Where do they get these numbers?

T.RowePrice®
INVEST WITH CONFIDENCE

Help sustain your assets:
BE BULLISH IN RETIREMENT

A careful evaluation of your asset allocation and initial withdrawal amount in retirement is vital. The following tables show how stocks—in varying proportions—coupled with a realistic initial withdrawal amount could increase the probability of comfortably funding a 25-, 30-, or even 35-year retirement.

For example, the middle table suggests that there is an 80% chance that a mix of 40% stocks and 60% bonds would sustain a 4% initial withdrawal amount (increased 3% annually for inflation) throughout a 30-year retirement.

25-YEAR RETIREMENT

Initial Withdrawal Amount	Stock/Bond* Mix 100/0	80/20	60/40	40/60	20/80
3%	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5	79	77	77	69	58
6	54	52	46	36	18
7	39	34	26	13	3
8	27	21	13	4	0

30-YEAR RETIREMENT

Initial Withdrawal Amount	Stock/Bond* Mix 100/0	80/20	60/40	40/60	20/80
3%	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4	77	79	[illegible]	[illegible]	74
5	60	60	56	46	29
6	44	43	33	13	5
7	31	25	15	6	2
8	20	14	7	1	0

35-YEAR RETIREMENT

Initial Withdrawal Amount	Stock/Bond* Mix 100/0	80/20	60/40	40/60	20/80
3%	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4	70	71	70	60	52
5	53	51	41	32	13
6	38	33	23	11	1
7	28	19	11	3	0
8	16	11	4	0	0

Key: More Likely — Less Likely**

Armed with the above, investors can customize an asset mix and initial withdrawal amount to coincide with their specific requirements.

**The likelihood of having at least $1 remaining in the portfolio at the end of the retirement period.

For more information, please call your investment professional or T. Rowe Price at 1-855-829-6343.

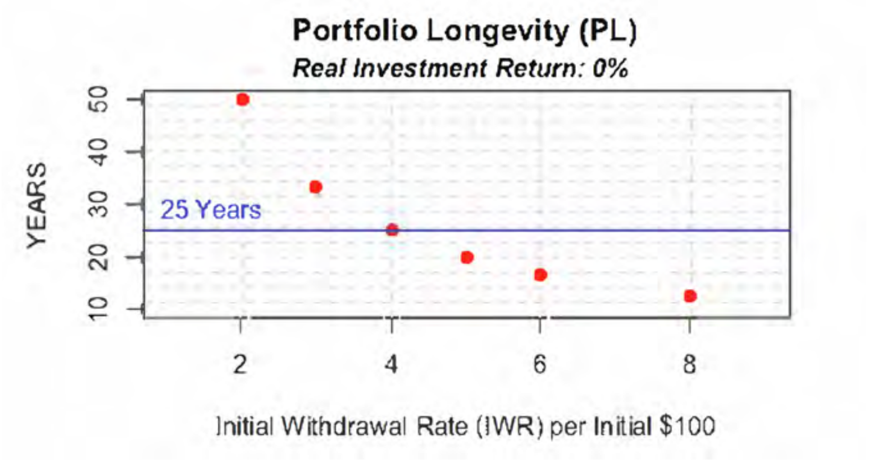
Portfolio Longevity: Agenda and Plan

1. What is Portfolio Longevity (PL) under *fixed* returns and withdrawals?
2. What is PL under *stochastic* (i.e. random) returns?
3. What are the drivers of the variability of PL?
4. How does this relate to the (infamous) 4% rule?
5. How can you *extend* the longevity of a portfolio?

Fixed Spending and a Systematic Withdrawal Plan (SWiP)

- You start retirement with a *nest egg* (a.k.a. financial capital, a.k.a. investment portfolio) of exactly $M = \$1,000,000$ and plan to withdraw exactly $w = \$50,000$ (in real terms) at the end of every year until the money runs-out. For now, ignore income taxes.
- If your money is invested in a simple bank account earning $v = 0\%$ (i.e. zero) interest, then it will last for exactly $M/w = 20$ years. At the end of the first year you will have $\$950,000$ remaining, at the end of the second year it will be $\$900,000$, etc., all the way to zero (at the end of) year twenty. This is trivial.

PL = M x (1/w)



Portfolio Longevity (PL)
Real Investment Return: 0%
YEARS
50
40
30
20
10
25 Years
2
4
6
8
Initial Withdrawal Rate (IWR) per Initial $100

Defining Portfolio Longevity with Interest

What if $v > 0$? The funds should last for more than $M/w = 20$ years. But how much longer, exactly? What is the longevity of a portfolio earning a fixed interest rate of v, subjected to a fixed initial withdrawal rate of w/M? Note. It could be infinite if v is high enough.

Computing (simple) Portfolio Longevity

- If portfolio longevity (measured in years) is denoted by L, and both withdrawals $w > 0$ and investment earnings $v > 0$ take place continuously in time, then L will satisfy the following equation:

$$L = \frac{1}{v} \ln \left[\frac{w/M}{w/M - v} \right], \tag{1}$$

  where ln[.] denotes the natural logarithm, and provided that $w/M > v$, otherwise $L := \infty$.

- Contrast with $L = M/w$, in the very simple case. Bonus Question: Prove convergence when $v \to 0$.

Examples of L

- For example, if $M = \$100$, the annual withdrawal rate is $w = 5$ and the continuously compounded interest (investment) rate is $v = 4\%$, portfolio longevity is $L = (1/0.04)\ln[5/(5-4)] = 40.236$ years. Compare this to (only) 20 years, when the interest rate is zero.
- If the withdrawal rate is increased to \$7 per year, portfolio longevity L drops by almost half to $L = (1/0.04)\ln[7/(7-4)] = 21.182$ years.
- Think about it this way. The present value of \$7 at a valuation rate of 4% for a period of 21.182 years is exactly \$100.
- So, you can use a standard business calculator to obtain the same result. Input $PV = 100$, an interest rate of $e^{0.05} - 1$ and cash-flow of -7 and the result should be (approximately) 21 years. Remember that it's *approximate* because equation (1) assumes continuous cash-flows.

Back to R-studio

Let's code-up this formula

Create Script

```
PL<-function(v,w,M)
{
if (w/M <= v) {999}
else {
  if (v==0){M/w}
  else (1/v)*log( (w/M)/(w/M-v) )
      }
}
```

Note the two if statement conditions. Pause and make sure you understand the logic. Warning: 999 is completely arbitrary. The proper answer is infinity (since the log is undefined.) I coded this way purely for (numerical) convenience, which will become clear in a moment.

Numerical Examples of Portfolio Longevity: Spending $4

Command Line

```
> PL(0,4,100)
[1] 25
> PL(0.01,4,100)
[1] 28.76821
> PL(0.02,4,100)
[1] 34.65736
> PL(0.03,4,100)
[1] 46.20981
> PL(0.035,4,100)
[1] 59.41262
> PL(0.0395,4,100)
[1] 110.9374
> PL(0.04,4,100)
[1] 999
```

Warning: PL can shrink from natural causes

If you earn negative (real) returns on your investments ($v < 0$), then your portfolio will shrink due to withdrawals as well as (poor) performance

Command Line

```
> PL(0,4,100)
[1] 25
> PL(-0.01,4,100)
[1] 22.31436
> PL(-0.02,4,100)
[1] 20.27326
```

Remember (again) the output is measured and reported in years.

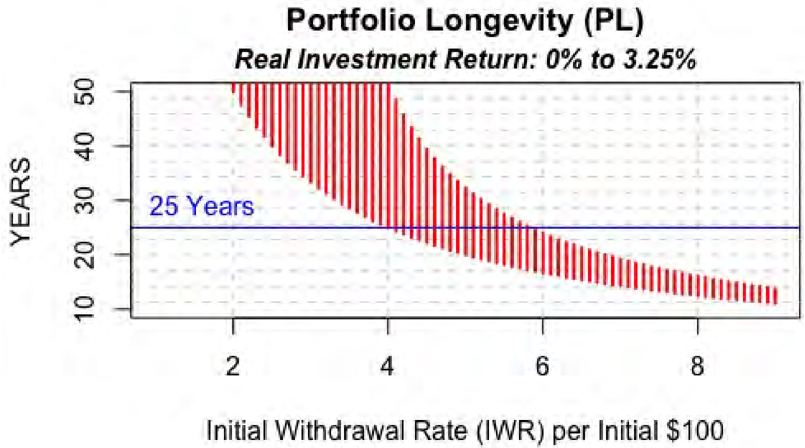
Generating (pretty) longevity pictures

Script

```
plot(c(1,9),c(10,50),type="n",
xlab="Initial Withdrawal Rate per 100 Dollar",ylab="YEARS")
title("Portfolio Longevity (PL)")
mtext("Real Return: 0 to 3.25 pct", side=3, line=0.3,font=4)
grid(ny=15,lty=20)
for (i in 10:90){
y1<-PL(0.0,i/10,100)
y2<-PL(0.0325,i/10,100)
segments(i/10,y1,i/10,y2,lwd=2,col="red")
}
abline(h=25,col="blue")
text(1.6,29,"25 Years",col="blue")
```

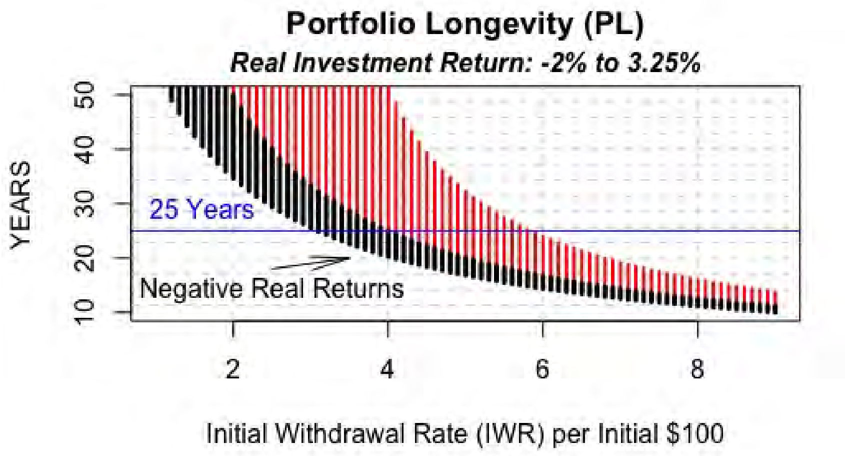
Note the (new) `segments` and `abline` command and what it does.

What happens as you increase initial withdrawal rate?



Visualizing it with investment-driven shrinkage

Slightly different coding for the picture. Add another segment



Things to Notice and Remember and Think About

- Notice how PL declines with withdrawal rate (remember $1/w$ from the $v = 0$ case) but at larger values the decline isn't as steep and the investment rate v doesn't make a very big difference.
- At a relatively lower withdrawal rate w, the PL value can be infinite (technically 999 in our code) and declines quite rapidly. Also, the investment rate v (range) makes a bigger difference.
- Oddly enough, the less you (decide to) withdraw, the more important the investment assumption (v) becomes in determining portfolio longevity.

The Justification (and some Calculus)

If portfolio longevity is L years, then by definition the present value (PV) of the withdrawals w, must be equal to the initial value of the portfolio M. In continuous time this implies that:

$$M \;=\; \int_0^L w\, e^{-vt}\, dt \;=\; \frac{w}{v}\left(1 - e^{-vL}\right). \tag{2}$$

This leads to:

$$e^{-vL} \;=\; \left(\frac{w - Mv}{w}\right) \;\Longrightarrow\; L \;=\; \frac{1}{v} \ln\left[\frac{w/M}{w/M - v}\right], \tag{3}$$

as long as $w/M > v$, which allows us to take logs of both sides. Note the derivative of longevity L, w.r.t. w, is:

$$\frac{\partial L}{\partial w} \;=\; -\left(\frac{1}{w^2/M - v}\right), \tag{4}$$

and is (obviously) negative. In fact, when $w > 10$ (approximately, for $M = 100$) the derivative is less than one and increasing withdrawals by \$1 (only) reduces portfolio longevity by less than a year.

But investment return v isn't constant in real life....

- In the first year the portfolio might earn $\tilde{v}_1 = 10\%$, while in the second year it might earn $\tilde{v}_2 = 1\%$ and in the third year it might earn $\tilde{v}_3 = -5\%$, which means that the portfolio declines even without (w) withdrawals.
- In this (real life) situation the longevity of the portfolio is random or unknown in advance and critically depends on the future (unknown) portfolio investment returns.
- While it's impossible to pinpoint the exact date at which the portfolio will be exhausted, it's possible to analyze the statistical distribution of portfolio longevity via simulation techniques. **R is well-suited for this task.**

Generating Random Numbers for Investment Returns

Generally speaking there are two approaches:

\# 1 Collect a long enough series of historical financial returns (e.g. index funds, mutual funds or ETFs) and then scramble *a.k.a. bootstrap* them. Think of a big urn with numbers written on pieces of paper. You draw them one out at a time (and replace).

\# 2 Make some (simple) forward-looking analytic distribution assumptions and generate random numbers probabilistically.

Initially we will use the analytic distribution [# 2] methodology.

Here is an example or some (really) random numbers.

Generating 25 Random (Normal) Investment Returns in R

Command Line

```
> v<-rnorm(25,0.03,0.20)
> v
 [1]  0.001470724 -0.049536054  0.081968218  0.007595034
 [5] -0.004212151  0.374103915  0.014192556  0.034780243
 [9] -0.267013480  0.321024066  0.246950585 -0.141261295
[13] -0.114878223  0.168271758  0.101573676 -0.142078856
[17]  0.178961316 -0.151984803  0.254120678  0.082672077
[21]  0.124891360  0.159017064  0.342147262  0.481518068
[25]  0.141713450
> summary(v)
     Min.   1st Qu.    Median      Mean   3rd Qu.      Max.
-0.267014 -0.004212  0.082672  0.089840  0.178961  0.481518
```

Notice that I generated (i.e. requested) a mean return of 3%, but got a sample mean of 8.9%. That is random life!

Let's do 25 again

Command Line

```
> v<-rnorm(25,0.03,0.20)
> v
 [1] -0.323782143 -0.286695245  0.137398657 -0.119615313
 [5]  0.130414993 -0.001835649 -0.139307787 -0.061605544
 [9]  0.201013955  0.077095585  0.139232315  0.107266139
[13]  0.002080820  0.007798052  0.225051697  0.350994380
[17]  0.169741813 -0.045779510 -0.296901387  0.187466617
[21]  0.130503545  0.028588365  0.316312497  0.036454973
[25] -0.223480817
> summary(v)
    Min.  1st Qu.   Median     Mean  3rd Qu.     Max.
-0.32378 -0.06161  0.03646  0.02994  0.13923  0.35099
>
```

Now the sample mean is 2.99%, which is what I had wanted.

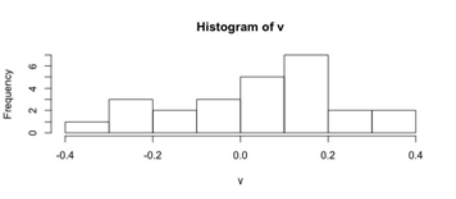
A Histogram of Your Investment Returns.

Enter this at the command line...

```
> hist(v)
```

This is the command for generating a histogram plot or figure.

...and you get this:



Not quite a *normal curve*, but with 25 data points what do you expect?

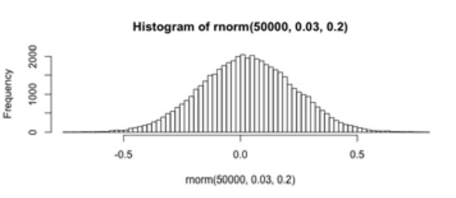
Do it again with 50,000 numbers

Enter this at the command line...

```
hist(rnorm(50000,0.03,0.20),breaks=75,xlim=c(-0.75,0.75))
```

This will simulate 50,000 random numbers that are normally distributed with a mean (average) value of 3% and standard deviation of 20%. Plot the histogram with 75 bins (or breaks).

...and you get this:



This is (much) nicer! And, it looks normal.

Simulated Investment Returns and Portfolio Longevity

The general idea is as follows:

- Start with an initial portfolio value: $M_0 = M$, and a desired (fixed) withdrawal rate: w.
- Simulate a (random, normal) continuously compounded investment return for the first year: $\tilde{v}_1$.
- The portfolio value at the end of the first year is $M_1 = (M_0 e^{\tilde{v}_1}) - w$.
- Generate a random return (again) and continue the process for the second year: $M_2 = (M_1 e^{\tilde{v}_2}) - w$, moving forward year-by-year.
- Generally, $M_j = (M_{j-1} e^{\tilde{v}_i}) - w$, is defined recursively.
- But, if-and-when the value of $M_j <= 0$, stop after j periods, and consider that **one sample** of your portfolio longevity $L = j$.
- Start another run with $M_0 = M$ and get a different value of L. Generate as many $L(i)$ values as you want. Do some statistics.

Generating PL Sample Paths

We are ready to design our first so-called **Monte Carlo simulation.**

Run Script and Create New Function

```
PLSM<-function(M,w,nu,sigma,N)
{
  path<-matrix(nrow=N,ncol=100)
  L<-c()
  for (i in 1:N){
    return<-exp(rnorm(100,nu,sigma))
    path[i,1]<-M
          for (j in 2:100){
          path[i,j]<-path[i,j-1]*return[j]-w
          if (path[i,j]<=0) {break}
                          }
             L[i]=j}
L}
```

Make sure you understand the code...Play with it.

Let's generate some numbers and then go back to understand the structure of the simulation in R.

Command Line

```
> PLSM(100,7,0.03,0.20,50)
 [1]  35  37  14  16  16  19  14  12 100  10  21  31  12  16
[17]  15  13  27  29  88  16  11  17  62  36  25 100  17  15
[33]  39  11  19  12  16  19 100  14 100  15  27  24  48  17
[49]  14  17
```

What are you looking at? (Remember. This is a simulation so your values will be different than mine, unless we use the same initial seed for the random numbers.)

More Sample Paths and Summary Statistics

Another run with $w = 5$ and some summary statistics.

Command Line

```
> sample1<-PLSM(100,5,0.03,0.20,5000)
> summary(sample1)
   Min.  1st Qu.  Median    Mean   3rd Qu.     Max.
    7.0     20.0    30.0    44.6    66.0      100.0
> length(sample1[sample1==100])/5000
[1] 0.2046
```

What are you looking at? What does it mean? Stop. Pause. Discuss.

Discussion and a Promise

- Later on I'll discuss where and how I obtained and estimated the specific ν and σ values. But basically they are fitted to historical values of real (after-inflation, after-fees) stock returns. You might have done this in an investment course.
- As you can see in this particular simulation run the smallest (minimum) value for the portfolio's longevity L was 7 years (ouch!) and the maximum value was 100 years. Remember, the 100 is artificial. Think: long enough.
- Notice median longevity was exactly 30 years. That is to say 50% of the $N = 5,000$ simulations, or $2,500$ scenarios exhibited longevity of less than or equal to 30 years and $2,500$ scenarios exhibited longevity of more than 30 years (good).
- The 1st quartile which was 20 years and the 3rd quartile which was 66 years. Stated differently, 50% of the portfolio longevity outcomes fell between 20 and 66 years, which is a range of 46 years.

Continue Discussion

- In contrast to the *median* value of 30 years, the *mean* value was higher and equal to 44.6 years. That is an extra 14.6 years compared to the *median* and might seem odd or confusing at first glance.
- The mean value is (highly) influenced by large outliers in a random sample. In this case the many scenarios with 100 years as the portfolio longevity, skewed the calculation of the mean towards a higher number. Technically speaking, the expected or mean longevity could be infinite. More on this later.
- In this particular simulation run, 20.46% of scenarios resulted in portfolio longevity values that were equal to or exceeded 100.
- Stated from the opposite perspective, in 79.54% of scenarios the portfolio was exhausted or depleted prior to the 100-year mark. It's very important to emphasize that this number depends critically on the initial withdrawal rate w as well as the portfolio investment parameters (ν, σ).

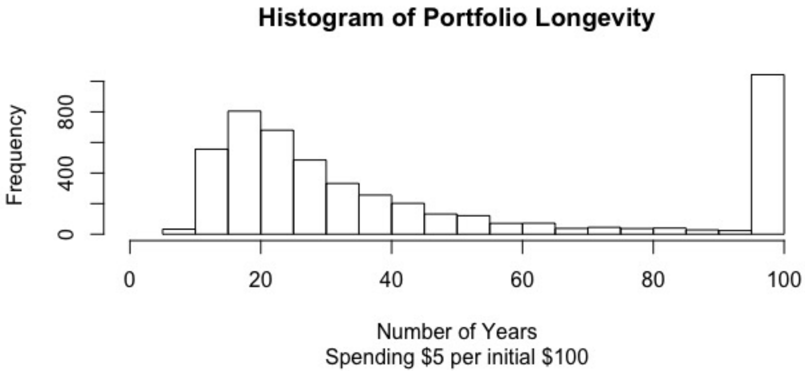
Generate a Histogram

Previously we generated a histogram for the investment returns, and now we generate a histogram for the portfolio longevity metric.

Command Line or Script

```
hist(sample1,main="Histogram of Portfolio Longevity",
xlab="Number of Years",
sub="Spending $5 per initial $100",
xlim=c(0,100))
```

Histogram of sample1: 5000 paths



Retirement Income Insights

- Notice the bimodal nature of the curve. There are a cluster of numbers around the value of 15 to 25, and then the density starts to decay very rapidly towards zero, but then spikes again at the value of 100.
- Remember the 100 number was **completely artificial** It is when I stopped the run and truncated the scenarios, but the basic intuitive point should be rather obvious from the picture and can be stated as follows.
- *You will either run out of money within four decades, or the money will last for ever...*
- Notice that there are very few intermediate scenarios or results. The histogram is most definitely not uniform or flat. Stated even more bluntly, the bad times happen in one of two very distinct times: soon or never.

More testing: Focus on Initial Withdrawal Rate (IWR)

```
> L4<-PLSM(100,4,0.035,0.20,5000)
> L5<-PLSM(100,5,0.035,0.20,5000)
> L6<-PLSM(100,6,0.035,0.20,5000)
> L7<-PLSM(100,7,0.035,0.20,5000)
> summary(L4)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   9.00   26.00   50.00   59.85  100.00  100.00
> summary(L5)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   7.00   21.00   34.00   50.03  100.00  100.00
> summary(L6)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   7.00   17.00   25.00   40.44   53.00  100.00
> summary(L7)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   7.00   15.00   20.00   32.34   34.00  100.00
```

Does the Money Run Out within 30 years?

For what fraction of the 5,000 simulation paths was the Portfolio Longevity (PL) less than 30 years?

Command Line

```
> length(L4[L4<=30])/5000
[1] 0.3214
> length(L5[L5<=30])/5000
[1] 0.4562
> length(L6[L6<=30])/5000
[1] 0.5962
> length(L7[L7<=30])/5000
[1] 0.7118
```

Intuitively, the higher the initial withdrawal rate (IWR) the higher the probability that the money doesn't last for 30 years (of retirement.)

Questions and Problems

- Create vectors of 25000 portfolio longevity values, assuming withdrawal rates of $w = 3, 4, 5, 6, 7$ and investment returns of $\nu = 1\%, \sigma = 8\%$ (low risk portfolio) as well as $\nu = 3.5\%, \sigma = 17\%$ (higher risk portfolio.) These are: 25000 x 5 x 2 numbers in total.
- Compute the median portfolio longevity for each of the 10 cases, as well as the 1st and 3rd quantile.
- For each of the 10 cases compute the fraction of the 25000 simulated values that are less than 30 years.
- Discuss the results, proper withdrawal rates and whether it is better to have a safer ($\nu = 1\%, \sigma = 8\%$) or riskier ($\nu = 3.5\%, \sigma = 17\%$) asset allocation as it relates to portfolio longevity.

Break: Deeper Dive into Portfolio Longevity:

1. What is the so-called Sequence of Returns (SoR) Effect?
2. How do we measure (and monitor) SoR Risk
3. How can SoR be controlled and PL extended?

Easy Quiz: You have $100,000

- Invest and hold $100,000 in a fund earning:

  +27% in year #1;

  +7% in year #2;

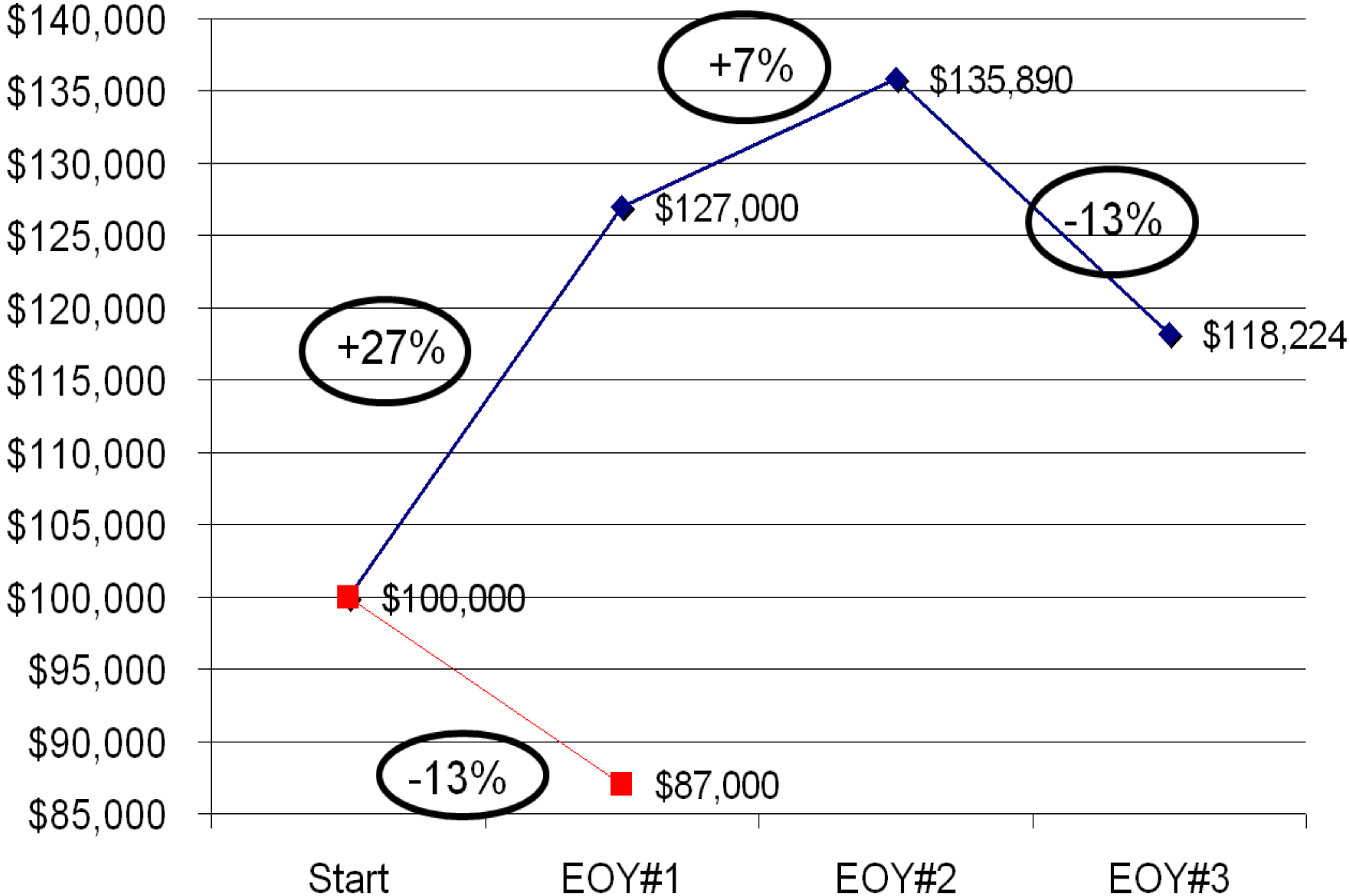
  -13% in year #3.

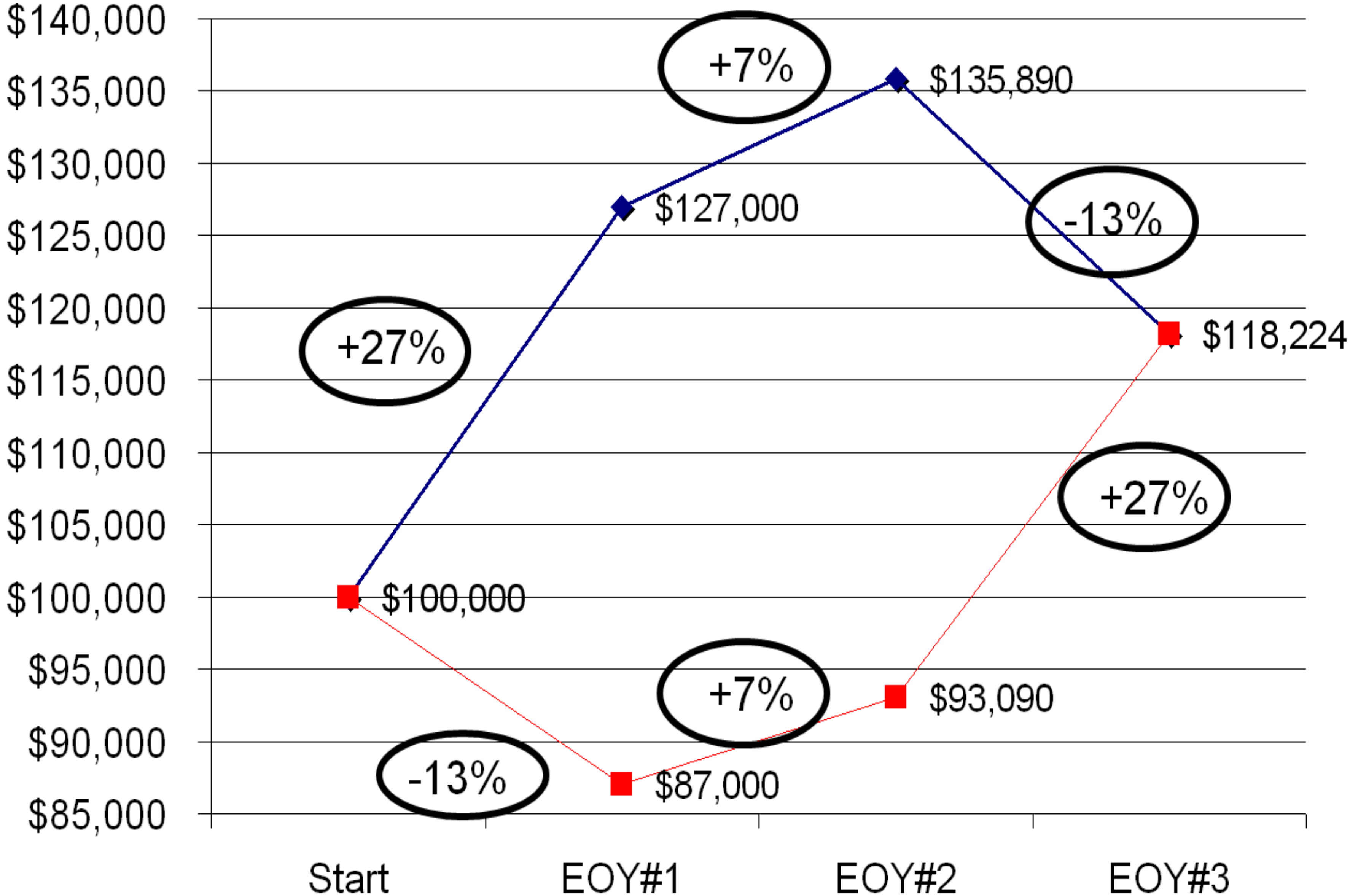
Question: Are you ahead after 3 years?


$140,000
$135,000
$130,000
$125,000
$120,000
$115,000
$110,000
$105,000
$100,000
$95,000
$90,000
$85,000
+27%
+7%
-13%
$100,000
$127,000
$135,890
$118,224
Start
EOY#1
EOY#2
EOY#3

What happens if I change the order of returns?

- Reverse the **sequence of investment returns.**
- Earn **(-13%)** during the first year, **(+7%)** in the second year and **(+27%)** in the third year.
- Start with same $100,000
- Do you end-up with more/less than **$118,224**?


$140,000
$135,000
$130,000
$125,000
$120,000
$115,000
$110,000
$105,000
$100,000
$95,000
$90,000
$85,000
+7%
$135,890
$127,000
-13%
$118,224
+27%
$100,000
-13%
$87,000
Start
EOY#1
EOY#2
EOY#3



$140,000
$135,000
$130,000
$125,000
$120,000
$115,000
$110,000
$105,000
$100,000
$95,000
$90,000
$85,000
+7%
$135,890
$127,000
-13%
+27%
$118,224
+27%
$100,000
+7%
$93,090
-13%
$87,000
Start
EOY#1
EOY#2
EOY#3

Why?

\$100,000 x (1.27) x (1.07) x (0.87) =

\$100,000 x (0.87) x (1.07) x (1.27) =

\$100,000 x (1.07) x (0.87) x (1.27) =

\$100,000 x (1.0574)^3 =

\$118,224

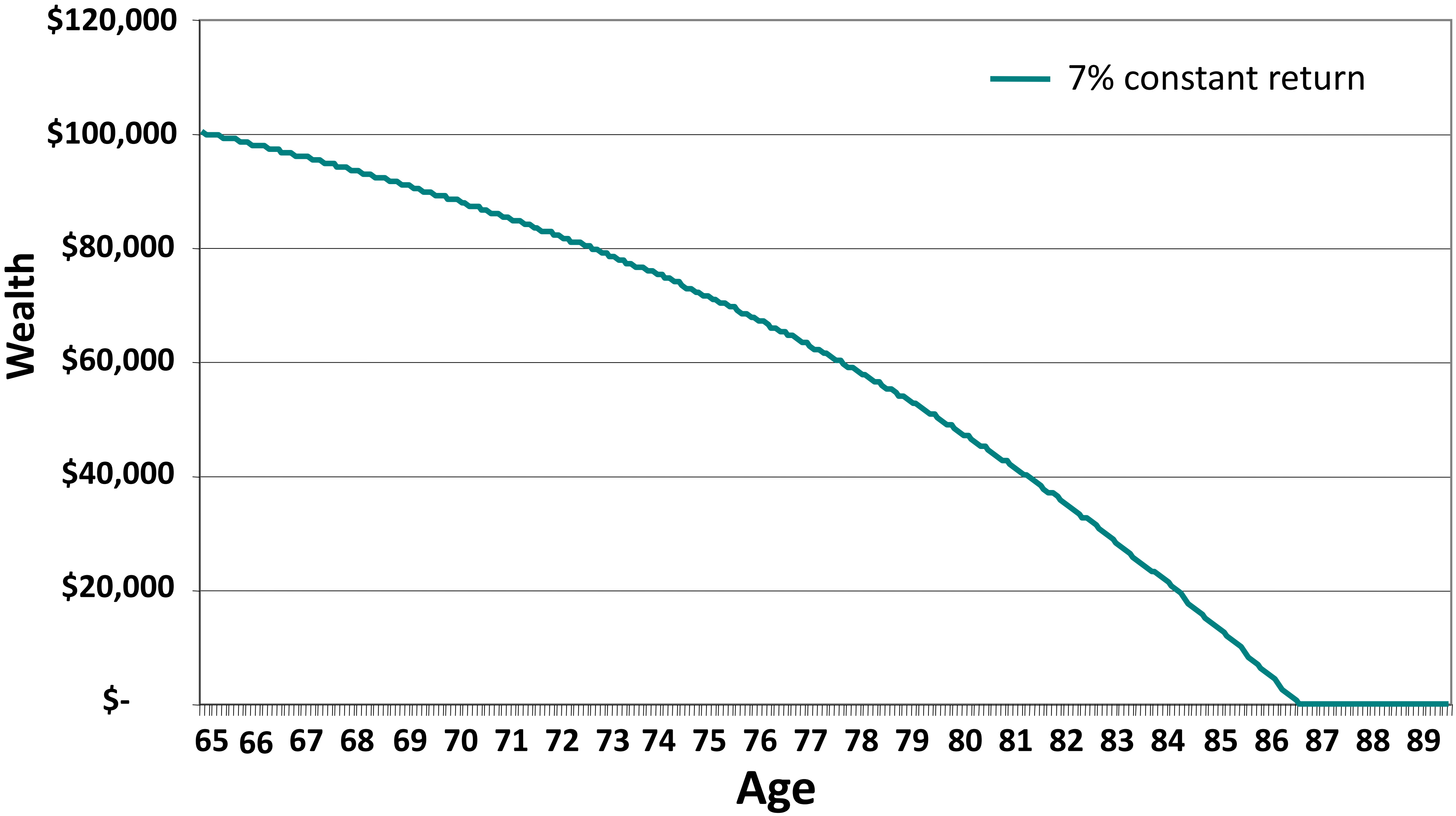
Retirement Income Case Study

Age 65 retirement wealth:	$100,000
Desired monthly income (real):	$750 (= $9,000/yr)
Assumed investment return:	Constant APR of 7%

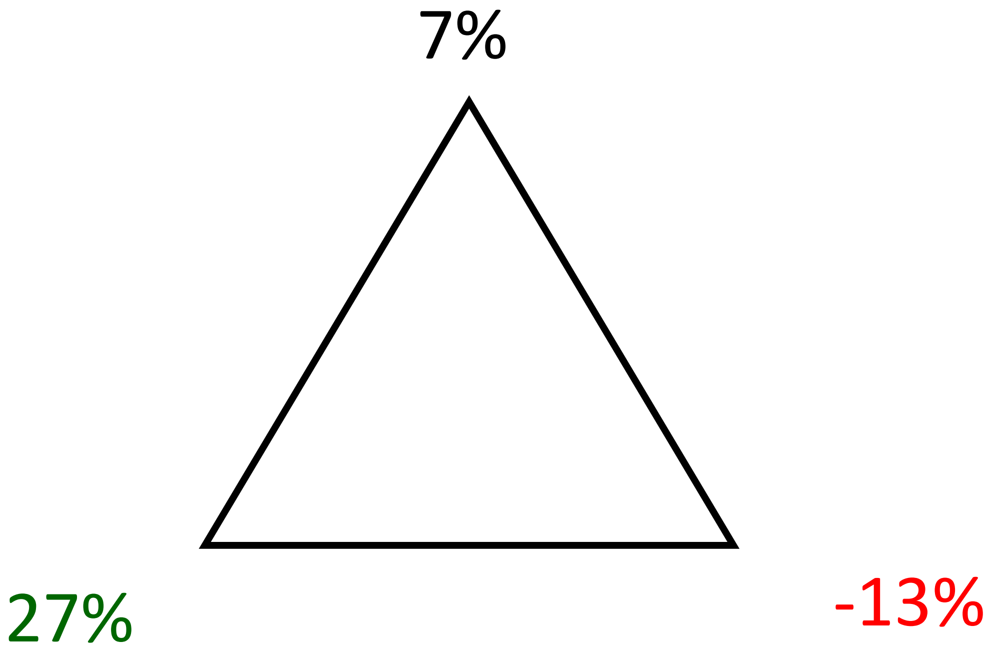
(= 0.58% monthly)

What is the longevity of your portfolio?

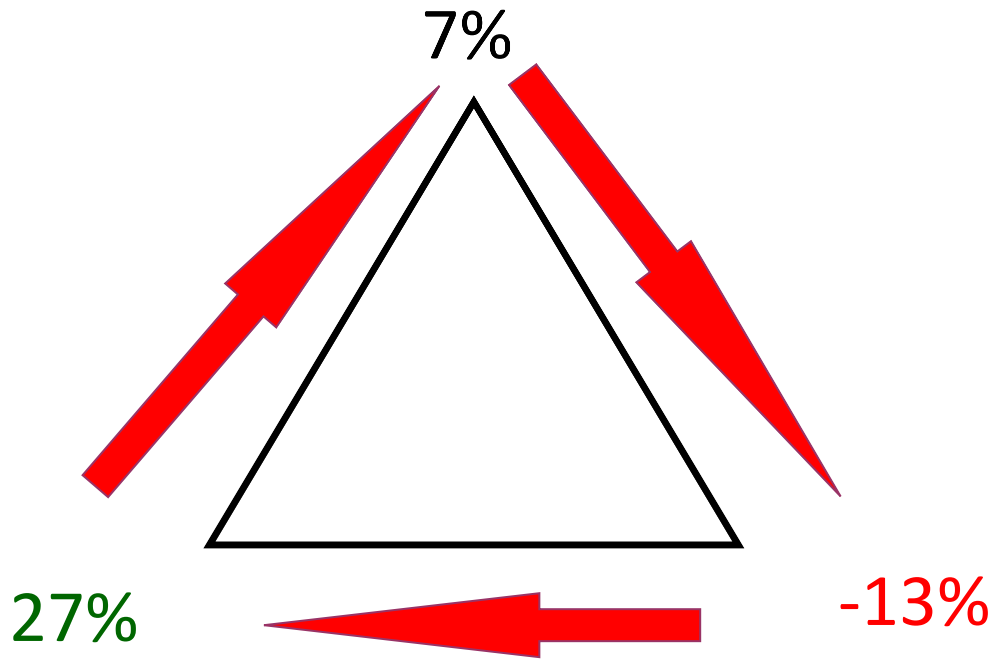
Numerical Example:
Spend $750 per month and earn 7% per year….



Simple Randomness:

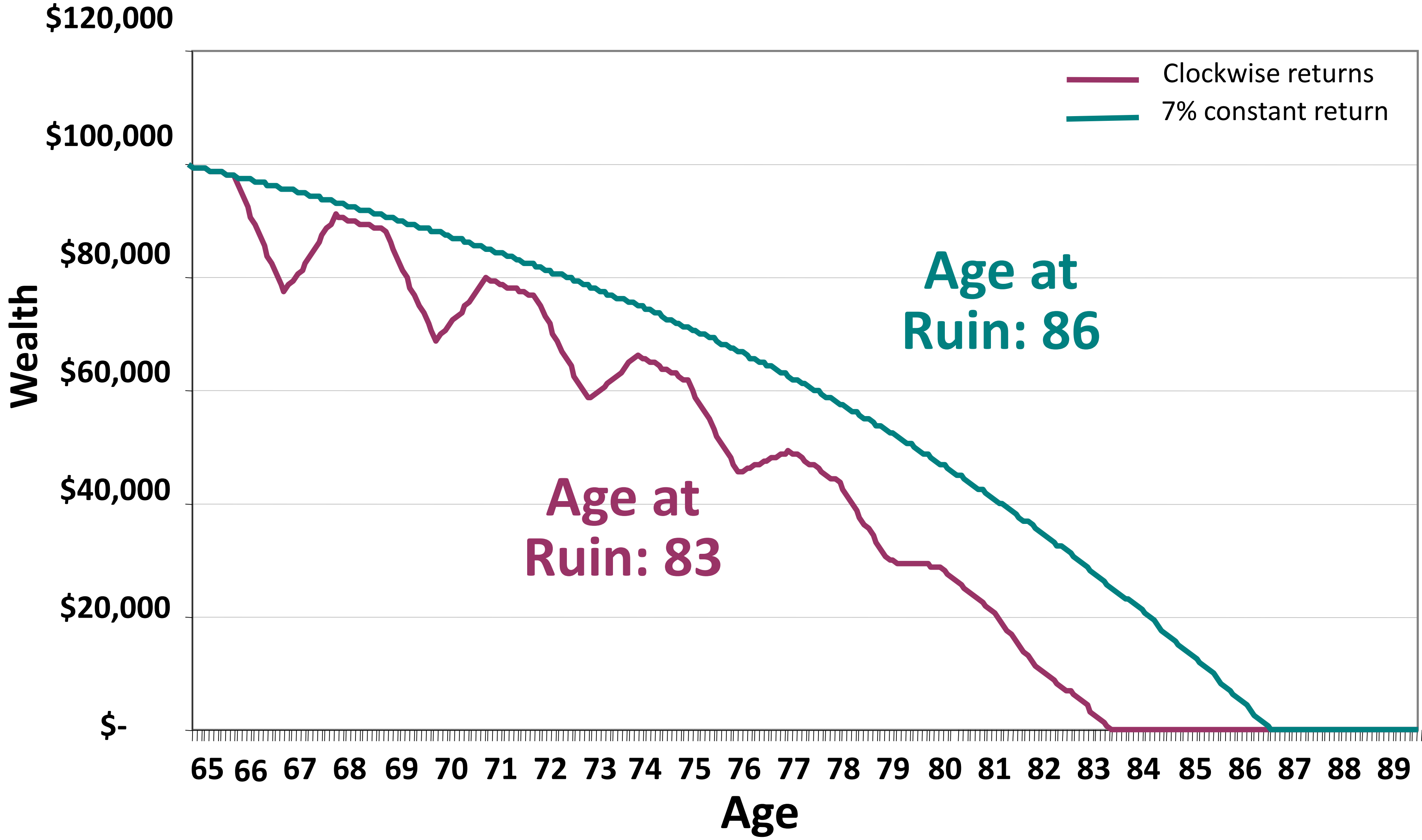


Simple Randomness:

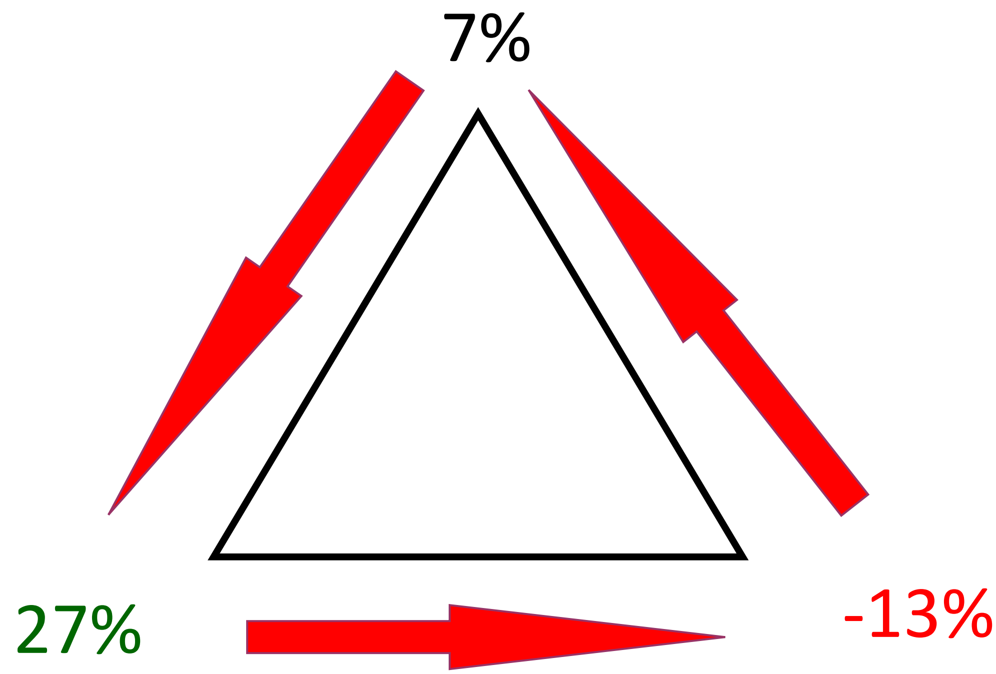


+7%, -13%, +27%, +7%, -13%, +27%, +7%, -13%, +27%....

Clockwise Random Returns

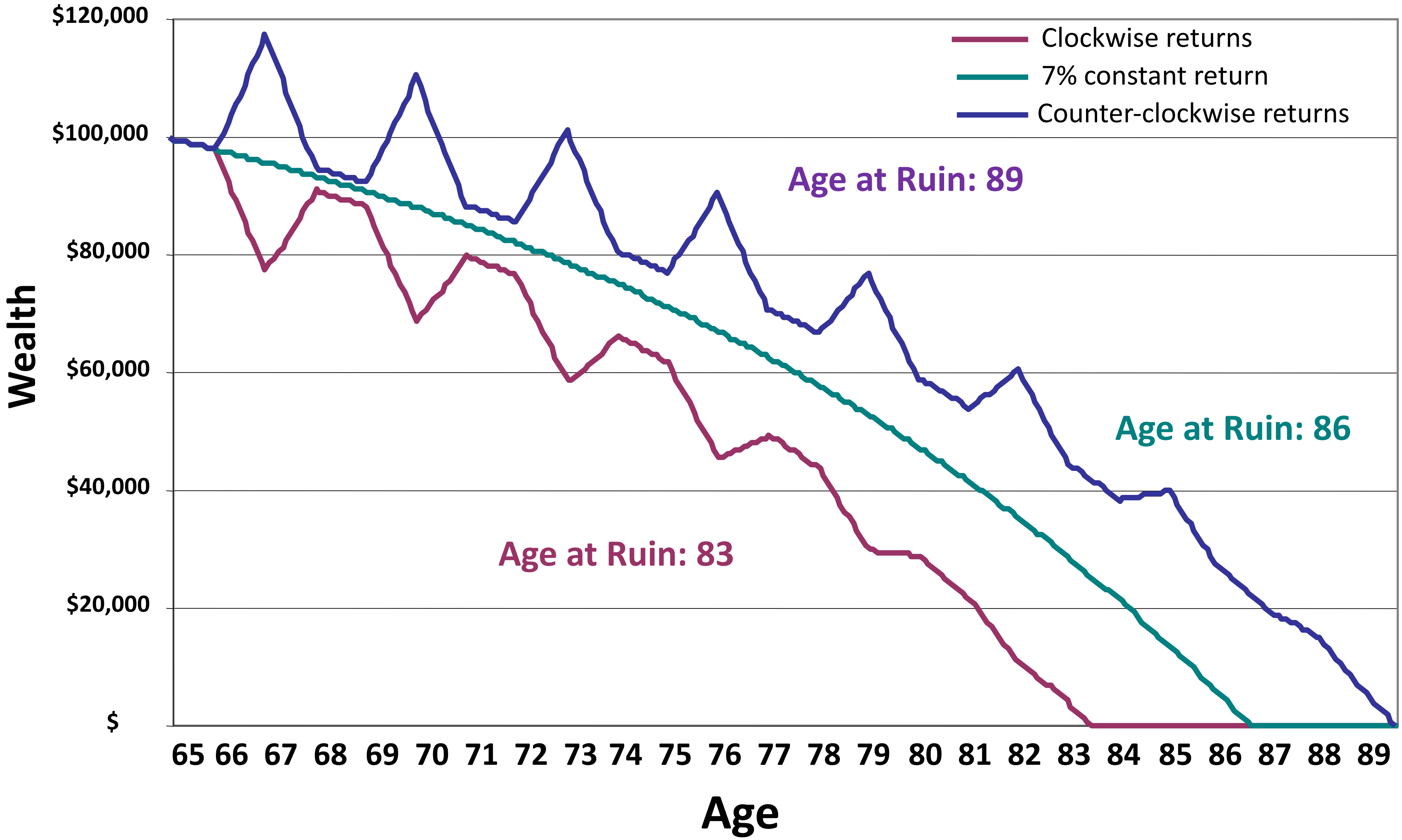


Reversal of Return Sequence: Counter-Clockwise Returns

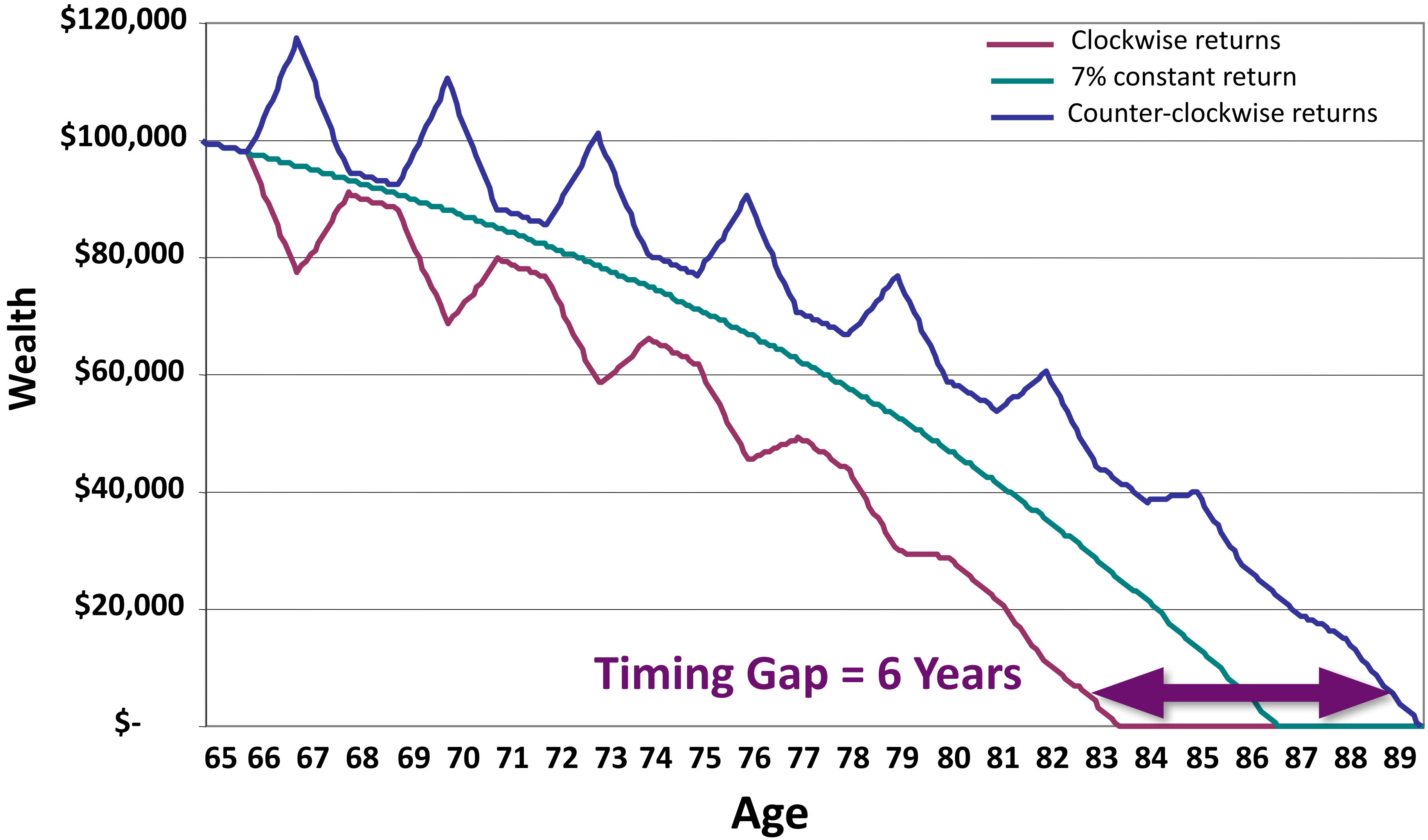


+7%, +27%, -13%, +7%, +27%, -13%, +7%, +27%, -13%....

Counter-Clockwise Returns



Counter-Clockwise Returns



SoR Risk: Summary Table

Return Sequence	***Ruin Age***	***+/- Months***
+7%, +7%, +7%...	86.50	
+7%, -13%, +27%...	83.33	**-38**
+7%, +27%, -13%...	89.50	**+36**

***Assumes $9,000 spending per year.**

SoR Risk: Summary Table

Return Sequence	*Ruin Age*	*+/- Months*
+7%, +7%, +7%...	86.50	
+7%, -13%, +27%...	83.33	**-38**
+7%, +27%, -13%...	89.50	**+36**
-13%, +7%, +27%...	81.08	**-65**
+27%, +7%, -13%...	94.92	**+101**

***Assumes $9,000 spending per year.**

Sequence of Returns: Sample Videos

Google "sequence of returns" and watch some of the videos...

New Portfolio Longevity Function: Store More Information

Script

```
PLSM2<-function(M,w,nu,sigma,N){
  path<-matrix(nrow=N,ncol=100)
  L<-matrix(nrow=N,ncol=4)
  for (i in 1:N){
    return<-exp(rnorm(100,nu,sigma))
    L[i,1]<-prod(return[1:10])^(1/10)-1
    L[i,2]<-prod(return[11:20])^(1/10)-1
    L[i,3]<-prod(return[21:30])^(1/10)-1
    path[i,1]<-M
    for (j in 2:100){
      path[i,j]<-path[i,j-1]*return[j]-w
      if (path[i,j]<=0) {break}
                  }
    L[i,4]=j}
  L}
```

Explanation

We are (again) simulating portfolio longevity (PL), but in addition we are keeping track of the cumulative investment returns during the first three decades. We would like to investigate how `L[,1]`, `L[,2]` and `L[,3]` impact portfolio longevity. This will be investigated using regression (and correlation) techniques.

A Canonical Regression

We simulate 5,000 portfolio longevity numbers and analyze results.

Command Line

```
L<-PLSM2(100,5,0.045,0.18,5000)
DR1<-L[,1]
DR2<-L[,2]
DR3<-L[,3]
PL<-L[,4]
fit<-lm(PL~DR1+DR2+DR3)
summary(fit)
```

Note the new function `lm()` which we have not used before. It will in effect regress the portfolio longevity (PL) value on the returns from the first three decades.

And you should see (something like) this

```
Call:
lm(formula = PL ~ DR1 + DR2 + DR3)
Residuals:
    Min       1Q  Median       3Q      Max
-81.110 -15.634  -2.671  15.322  62.931
Coefficients:
            Estimate   Std. Error t value    Pr(>|t|)
(Intercept)  22.0942     0.5013   44.08      <2e-16 ***
DR1         396.4617     4.9232   80.53      <2e-16 ***
DR2         241.8925     4.7619   50.80      <2e-16 ***
DR3         131.0506     4.9539   26.45      <2e-16 ***
Residual standard error: 20.67 on 4996 degrees of freedom
Multiple R-squared:   0.66,Adjusted R-squared:  0.6597
F-statistic:  3232 on 3 and 4996 DF,  p-value: < 2.2e-16
```

Explanation

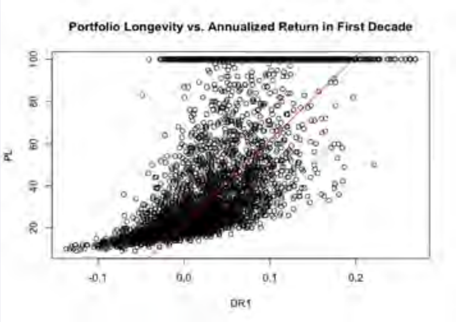
Let's pause to ensure we understand every single one of these (output) metrics and what they represent. Focus on the estimated coefficients on `DR1, DR2, DR3`, the intercept `22.09` and the R-squared values.

Impact of Decade Number One:

Command Line

```
plot(DR1,PL)
title("Portfolio Longevity vs.
        Annualized Return in First Decade")
abline(fit$coefficients[1],fit$coefficients[2],col="red")
```

...and you should see:

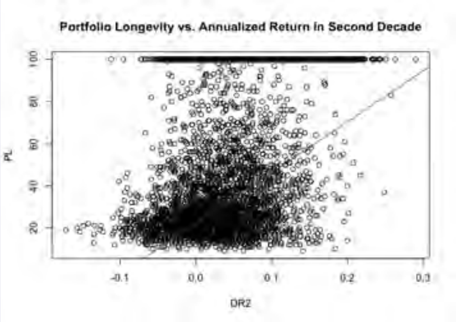


Impact of Decade Number Two:

Command Line

```
plot(DR2,PL)
title("Portfolio Longevity vs.
        Annualized Return in Second Decade")
abline(fit$coefficients[1],fit$coefficients[3],col="red")
```

...and you should see:

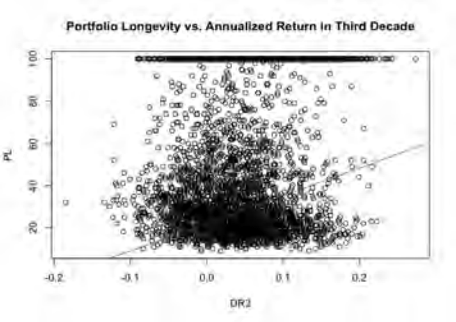


Impact of Decade Number Three:

Command Line

```
plot(DR3,PL)
title("Portfolio Longevity vs.
        Annualized Return in Third Decade")
abline(fit$coefficients[1],fit$coefficients[4],col="red")
```

...and you should see:



Explanation

Notice how the (linear) fit deteriorates over the decades. Although all three are statistically significant, the coefficient on the first decades' return is three times higher than the coefficient on the third decade's return.

Crude Regression: Different Setup

The dependent (left-hand) variable in the regression is set to a value of **zero** if the portfolio didn't survive for 30 years, and set to a value of **one** if the portfolio lasted for more than 30 year. By this construction the regression results are not skewed by the artificial (and arbitrary) stop at year 100. Yes,, it would be more accurate to run a logistic regression, but this (crude) set-up is a good start.

Command Line or Script

```
L<-PLSM2(100,5,0.045,0.18,5000)
DR1<-L[,1]
DR2<-L[,2]
DR3<-L[,3]
PL<-L[,4]
PL[PL<=30]<-0
PL[PL>30]<-1
```

Notice how we do the replacement in **R**.

First, Summary Statistics

The simulation is based on a spending rate of $w = 5$ per initial $M = 100$, a continuously compounded (mean) return of $\nu = 4.5\%$ and a standard deviation of $\sigma = 18\%$

Command Line

```
> mean(PL)
[1] 0.6526
> cor(PL,DR1)
[1] 0.616747
> cor(PL,DR2)
[1] 0.3559911
> cor(PL,DR3)
[1] 0.1060361
```

So, the probability the portfolio lasts for (over) 30 years is 65%, and the correlation with individual decade returns is 61.6% for the first decade, 35.6% for the second decade and 10.6% for the third decade.

Binary Regression

Now we run the regression with the exact same syntax as before.

Command Line

```
fit<-lm(PL~DR1+DR2+DR3)
summary(fit)
```

But this time the dependent variable *PL* is either zero or one. I stress (again) that the `glm` function would be more appropriate in general, but at this (early) stage and given our objective to measure the impact of various decades, I'll keep it simple.

And you should see (something like) this

```
Call:
lm(formula = PL ~ DR1 + DR2 + DR3)
Residuals:
     Min       1Q   Median       3Q      Max
-0.89795 -0.28036  0.02012  0.27439  0.70867
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.233121   0.007986   29.19   <2e-16 ***
DR1         4.952411   0.077837   63.62   <2e-16 ***
DR2         2.933001   0.078234   37.49   <2e-16 ***
DR3         0.910722   0.076669   11.88   <2e-16 ***
Residual standard error: 0.3283 on 4996 degrees of freedom
Multiple R-squared:  0.5251,Adjusted R-squared:  0.5248
F-statistic:  1841 on 3 and 4996 DF,  p-value: < 2.2e-16
```

Intepretation

Once again, all three coefficients are (highly) statistically significant, but notice that the first decade's coefficient contributes five-times more to the probability of reaching a 30-year longevity, compared to the third decade. These results do **not** depend on whether the (extreme) longevity value is set at 100 (or 50 or 1,000) years.

Simulation: Higher Spending and Lower Returns

Command Line or Script

```
L<-PLSM2(100,6,0.04,0.18,5000)
DR1<-L[,1]
DR2<-L[,2]
DR3<-L[,3]
PL<-L[,4]
PL[PL<=30]<-0
PL[PL>30]<-1
fit<-lm(PL~DR1+DR2+DR3)
summary(fit)
```

Notice that $w = 6$ and $\nu = 4\%$, which are slightly different from the parameters we used in the prior simulation.

Pay attention to the intercept and ratio of coefficients

```
Call:
lm(formula = PL ~ DR1 + DR2 + DR3)
Residuals:
     Min       1Q   Median       3Q      Max
-0.93691 -0.26342 -0.00922  0.26663  0.82033
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 0.065234   0.007424   8.787   <2e-16 ***
DR1         5.457368   0.078125  69.855   <2e-16 ***
DR2         2.990752   0.077874  38.405   <2e-16 ***
DR3         0.814428   0.077274  10.539   <2e-16 ***
Residual standard error: 0.3294 on 4996 degrees of freedom
Multiple R-squared:  0.564,Adjusted R-squared:  0.5638
F-statistic:  2155 on 3 and 4996 DF,  p-value: < 2.2e-16
```

Finally, Summary Statistics

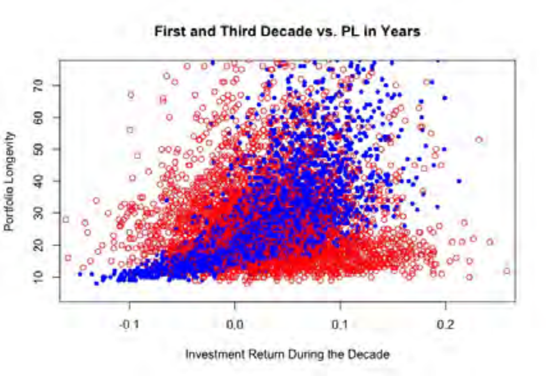
Command Line

```
> mean(PL)
[1] 0.4644
> cor(PL,DR1)
[1] 0.6525485
> cor(PL,DR2)
[1] 0.3543755
> cor(PL,DR3)
[1] 0.1114658
```

From a qualitative point of view it is similar to prior results, although notice that the first decade is (slightly) more impactful.

Final Picture: Spend \$6, $\nu = 4\%$, $\sigma = 18\%$

```
plot(c(-0.15,0.25),c(5,75),type="n",
for (i in 1:5000){
  points(DR1[i],PL[i],col="blue",pch=20)
  points(DR3[i],PL[i],col="red",pch=21)}
```



What is done to mitigate the impact of SoR?

The financial industry has been trying to design products that can extended the longevity of a portfolio by reducing the sequence of return (SoR) effect. This include variable annuities (VAs) with guaranteed living withdrawal benefits (GLWB) or *collared portfolios* using puts & calls.

Of course, this whole approach assumes that people will continue to spend the exact same amount of money during retirement regardless of how markets perform, which is ridiculous (i.e. sub-optimal) and something I'll return to in more advanced lectures.

At this (early) point the objective was to (i.) understand the so-called 4% rule, (ii.) measure the longevity of a portfolio, (iii.) simulate simple portfolio returns, (iv.) simulate random longevity, and finally, explain the phenomenon known as sequence of returns. **Q.E.D.**